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Two-dimensional periodic waves in supersonic flow of a Bose–Einstein condensate

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Abstract
Stationary periodic solutions of the two-dimensional Gross–Pitaevskii equation are obtained and analysed for different parameter values in the context of the problem of a supersonic flow of a Bose–Einstein condensate past an obstacle. The asymptotic connections with the corresponding periodic solutions of the Korteweg–de Vries and nonlinear Schrödinger equations are studied and typical spatial wave distributions are discussed.

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1. Introduction

The Gross–Pitaevskii (GP) equation plays a prominent role in the description of nonlinear dynamics of Bose–Einstein condensates (BEC) (see, e.g., [1]). It describes, in the so-called mean-field approximation, the behaviour of the order parameter $\psi(\mathbf{r})$ (the 'condensate wavefunction') and has the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(\mathbf{r})\psi + g|\psi|^2\psi, \tag{1}$$

where $V(\mathbf{r})$ denotes the potential of the external forces acting on the condensate (e.g. the confining trap potential), g is an effective coupling constant arising due to interatomic collisions with the s-wave scattering length a_s (positive for repulsive interactions), $g = 4\pi\hbar^2 a_s/m$, m being the atomic mass. The GP equation (1) takes into account the dispersive and nonlinear properties of the condensate which can give rise to various nonlinear structures in a BEC flow. In particular, vortices, bright (for $g < 0$) and dark (for $g > 0$)

solitons have been extensively studied theoretically in the framework of the GP equation and observed in experiments (see, e.g., [1] and references therein).

An important insight into the structure of the solutions of the GP equation is provided by the fact that its potential-free, one-dimensional reduction coincides with the integrable nonlinear Schrödinger equation

$$\mathrm{i}\psi_\tau + \psi_{XX} - 2\sigma|\psi|^2\psi = 0. \tag{2}$$

Here $X = x/\xi$, $\tau = tc_s/\xi$ and $\sigma = \operatorname{sgn} g$, where $\xi = \hbar/\sqrt{2mn_0 g}$ is the healing length and $c_s = \hbar/\sqrt{2}m\xi$ is the sound velocity in a BEC of density n_0.

In recent experiments [2] on free expansion of a non-rotating BEC after its release from a trap, the new interesting blast wave patterns have been observed. Using the modulation solutions of the one-dimensional NLS equation (2) (defocusing case), obtained earlier in [3–6], and numerical simulations in 2D and 3D, these blast waves have been identified in [7–9] with expanding dispersive shock waves, which represent oscillatory counterparts of classical evolutionary gas-dynamic shocks.

Yet another type of nonlinear wave patterns has been observed in another series of experiments on the flow of a non-rotating BEC past macroscopic obstacles also reported in [2]. In [10] these structures have been associated with spatial dispersive shock waves. Spatial dispersive shock waves represent dispersive analogues of the well-known viscous spatial shocks (oblique jumps of compression) occurring in supersonic flows of compressible fluids past obstacles. In a viscous fluid, the shock can be represented as a narrow region within which strong dissipation processes take place and the thermodynamic parameters of the flow undergo sharp change. In contrast, if viscosity is negligibly small compared with dispersion effects, the shock discontinuity resolves into an expanding in space oscillatory structure which transforms gradually, as the distance from the obstacle increases, into a 'fan' of stationary solitons. The theory of the generation of such spatial dispersive shock waves has been developed in [11] in quite general terms for supersonic dissipationless flows past slender body when the flow can be asymptotically described by the Korteweg–de Vries (KdV) equation and an effective description of dispersive shocks becomes possible via the Whitham modulation theory [12–14].

In a different approximation, for *highly* supersonic flow past slender body, for a certain range of parameters, the GP equation *asymptotically* reduces to the one-dimensional nonlinear Schrödinger (NLS) equation (2), albeit for a completely different set of independent variables [10]. As a result, the problem of stationary dispersive shock waves in this case can be treated by similar methods of Whitham's theory. It is essential that in all cited papers, the analytical advances in the description of the dispersive shock waves have become possible owing to a complete integrability of the KdV and 1D NLS equations resulting in the possibility of representing the associated modulation Whitham systems in Riemann diagonal form (see [15] and references therein), which dramatically simplifies further analysis.

However, in real experiments the obstacles cannot be treated as slender bodies and the flow is not highly supersonic, so the spatial dispersive shock waves must be considered in the framework of the *full*, non-approximated GP equation (1). In such a study, one inevitably faces the major obstruction to obtaining analytical solutions to an initial or boundary value problem because the multi-dimensional GP equation, even in the absence of the external forcing term, is a non-integrable system and powerful spectral methods (inverse scattering transform, finite-gap integration) are not available for it. However, the Whitham modulation approach still remains a possibility provided a minimal structure (availability of a certain number of conservation laws and a travelling periodic wave solution) is present [16].

In the framework of the Whitham modulation theory, a dispersive shock wave is considered as a modulated nonlinear periodic wave whose parameters change slowly on a scale about one wavelength and one period (see [13, 15] for instance). For a certain spatio-temporal domain, the dispersive shock represents a train of solitons well separated from each other. In the context of the BEC flow past obstacle a 2D spatial dispersive shock wave is asymptotically (far enough from the obstacle) represented as a 'fan' of spatial dark solitons. When the obstacle is not very large, only one soliton can be generated. This simplest case has been considered recently in [17] where exact spatial soliton solution of the GP equation (1) was found for condensate with a flow and it was shown by a numerical simulation that such 'oblique' solitons can be generated by a supersonic flow past obstacle. To extend this theory to 'multi-soliton' dispersive shock waves represented by a modulated periodic wave, it is necessary, first of all, to find stationary periodic solutions of the GP equation (1), and this is the main aim of the present paper. We also note that the periodic solutions of the GP equation could be important in other, than BEC, areas where the multi-dimensional NLS equation is used to model nonlinear wave propagation.

2. Periodic solution

The family of solutions of equation (1) describing stationary states of a BEC can be sought in the form

$$\psi(\mathbf{r}, t) = \sqrt{n(\mathbf{r})} \exp\left(\frac{\mathrm{i}}{\hbar} \int^{\mathbf{r}} \mathbf{u}(\mathbf{r}')\, \mathrm{d}\mathbf{r}'\right) \exp\left(-\frac{\mathrm{i}\mu}{\hbar} t\right), \tag{3}$$

where $n(\mathbf{r})$ is the density of atoms in the BEC, $\mathbf{u}(\mathbf{r})$ denotes its velocity field and μ is the chemical potential. It is convenient to introduce the dimensionless variables

$$\tilde{\mathbf{r}} = \mathbf{r}/\sqrt{2}\xi, \qquad \tilde{n} = n/n_0, \qquad \tilde{\mathbf{u}} = \mathbf{u}/c_s. \tag{4}$$

Substituting equation (3) into (1) and separating real and imaginary parts we obtain a system of equations for the density $n(x, y)$ and the two components of the velocity field $\mathbf{u} = (u(x, y), v(x, y))$,

$$\begin{aligned} &(nu)_x + (nv)_y = 0 \\ &uu_x + vu_y + n_x + \left(\frac{n_x^2 + n_y^2}{8n^2} - \frac{n_{xx} + n_{yy}}{4n}\right)_x = 0 \\ &uv_x + vv_y + n_y + \left(\frac{n_x^2 + n_y^2}{8n^2} - \frac{n_{xx} + n_{yy}}{4n}\right)_y = 0 \end{aligned} \tag{5}$$

where we have omitted tildes for convenience of notation.

One can see that the $(0 + 2)$ reduction (5) of the GP equation (1) is drastically different from its $(1 + 1)$ NLS reduction (2) (which can also be represented in a hydrodynamic form by the change of variables analogous to (3)—see section 4). Putting aside the subtle integrability aspects (equation (2) is a completely integrable system while there is no indication of integrability for the system (5)) we note that, first of all, the scalar system corresponding to the one-dimensional NLS equation (2) consists of *two* equations (for n and u) while the spatial $(0 + 2)$ case leads to *three* equations (5). As a result, the structure of the system (5) is significantly more complicated. This can already be seen by comparing the dispersionless limits of (2) and (5). Indeed, the dispersionless limit of one-dimensional NLS equation (2) coincides with the classical shallow-water equations (or classical gas-dynamic equations with the adiabatic index $\gamma = 2$) which are always hyperbolic. Contrastingly, the dispersionless limit of (5) coincides with *stationary two-dimensional* gas-dynamic ($\gamma = 2$) equations which

have a mixed elliptic-hyperbolic structure depending on the absolute value of the flow velocity. Thus, one can expect a considerable difference in the behaviour of the periodic solutions and their modulations in these two different reductions of the GP equation.

We look for the solution of the system (5) in the form of a 'travelling' wave

$$n = n(\theta), \qquad u = u(\theta), \qquad v = v(\theta), \tag{6}$$

where $\theta = x - ay$, a being the 'slope' parameter of the stationary wave (the wave crests lie on parallel lines with the slope a to y-axis). Under this *ansatz*, the first equation (5) gives at once

$$u - av = \frac{A}{n}, \tag{7}$$

where A is the integration constant, and the other two equations reduce to

$$n_\xi^2 - 2nn_{\xi\xi} + 2n^3 - \frac{2B}{1+a^2}n^2 + \frac{A^2}{1+a^2} = 0, \tag{8}$$

where B is another integration constant and we have introduced new independent variable

$$\xi = \frac{2\theta}{\sqrt{1+a^2}} = \frac{2(x-ay)}{\sqrt{1+a^2}}. \tag{9}$$

One can verify by a direct substitution that equation (8) has the first integral

$$n_\xi^2 = n^3 - \frac{2B}{1+a^2}n^2 - \frac{2C}{1+a^2}n - \frac{A^2}{1+a^2}, \tag{10}$$

where C is an arbitrary constant. Equation (10) has the well-known solution in terms of elliptic functions. To write it down, we denote the zeros of the polynomial on the right-hand side of equation (10) as p_1, p_2, p_3, so that

$$n_\xi^2 = (n - p_1)(p_2 - n)(p_3 - n), \qquad p_1 \leqslant p_2 \leqslant p_3, \tag{11}$$

and suppose that $n = p_1$ at $\xi = 0$. As a result, we obtain

$$n = p_1 + (p_2 - p_1)\mathrm{sn}^2(\sqrt{p_3 - p_1}\,\xi/2; m), \tag{12}$$

where $\mathrm{sn}(\theta; m)$ is the Jacobi elliptic sine and

$$m = \frac{p_2 - p_1}{p_3 - p_1} \tag{13}$$

is the modulus. The constants A, B, C are connected with the zeros p_1, p_2, p_3 by the relations

$$\begin{aligned} p_1 + p_2 + p_3 &= \frac{2B}{1+a^2} \\ p_1p_2 + p_1p_3 + p_2p_3 &= -\frac{2C}{1+a^2} \\ p_1p_2p_3 &= \frac{A^2}{1+a^2}. \end{aligned} \tag{14}$$

It is worth noting that the components (u, v) of the velocity field are not determined unambiguously by the constants p_1, p_2, p_3 and a. Indeed, if n is known, we have only one equation (7) for calculation of u and v. Another equation can be added, if we restrict ourselves to the consideration of potential flows by imposing the condition

$$u_y = v_x, \tag{15}$$

which is consistent with the system (5) (see [17]), and for a single-phase wave (6) yields at once

$$au + v = D. \tag{16}$$

Here D is an additional integral of 'motion' which owes its existence to the Bernoulli theorem for the system (5). Equation (16) implies that the same spatial periodic profile of the density (12) with the slope a can be supported by different potential velocity fields. If we fix u and v at some point, then the constant D becomes determined, as well as the velocity components everywhere. Just this situation occurs in the case of the soliton solution [17] where the velocity components are supposed to be known at $|x| \to \infty$. Indeed, considering $m = 1$ in (12) and assuming $n = 1, u = M =$ constant, where M is the Mach number and $v = 0$ as $|x| \to \infty$ we arrive at the oblique dark soliton solution obtained in [17],

$$n = 1 - \frac{1-q}{\cosh^2\left[\sqrt{1-q}(x-ay)/(1+a^2)\right]}, \tag{17}$$

where $q = M^2/(1+a^2)$ and the velocity components are given by

$$u = \frac{M(1+a^2 n)}{(1+a^2)n}, \qquad v = -\frac{aM(1-n)}{(1+a^2)n}. \tag{18}$$

We note that we are not concerned here with the stability of the obtained periodic solutions which calls for further nonlinear modulation analysis. The stability of the oblique dark solitons (17) for $M > 1$ was established in [17] numerically.

It is important now to investigate the behaviour of the obtained periodic solution for the parameter values corresponding to some physically interesting asymptotic reductions of the system (5).

3. Small-amplitude nonlinear periodic waves

As was indicated in [17], if we consider a flow of BEC corresponding to small deviations from a uniform and homogeneous supersonic flow with $n = 1, u = M, v = 0\,(M > 1)$, and make asymptotic expansions

$$\begin{aligned} n &= 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \cdots \\ u &= M + \varepsilon u_2 + \varepsilon^2 n_2 + \cdots \\ v &= \varepsilon v_1 + \varepsilon^2 v_2 + \cdots, \end{aligned} \tag{19}$$

where $\varepsilon \ll 1$ is a small parameter, then substitution of (19) into (5) followed by introduction of the scaled variables

$$\zeta = \varepsilon^{1/2}(x - Vy), \qquad \tau = \varepsilon^{3/2} y, \tag{20}$$

leads, according to the standard reductive perturbation method, to relations

$$u_1 = -\frac{n_1}{M}, \qquad v_1 = \frac{V}{M} n_1, \qquad V = \sqrt{M^2 - 1}, \tag{21}$$

where n_1 obeys the KdV equation

$$n_{1,\tau} - \frac{3M^2}{2\sqrt{M^2-1}} n_1 n_{1,\zeta} + \frac{M^4}{8\sqrt{M^2-1}} n_{1,\zeta\zeta\zeta} = 0. \tag{22}$$

Its periodic solution is well known (see, e.g. [15]) so that the density profile can be expressed after returning to original (x, y)-coordinates as

$$n = 1 - \tfrac{1}{2} M^2 \varepsilon(\lambda_3 - \lambda_1 - \lambda_2) + M^2 \varepsilon(\lambda_3 - \lambda_2)\mathrm{sn}^2\left[\sqrt{\lambda_3 - \lambda_1}\,\varepsilon^{1/2}(x - ay), m\right], \tag{23}$$

where

$$a = \sqrt{M^2 - 1} - \frac{\varepsilon(\lambda_1 + \lambda_2 + \lambda_3)M^2}{4\sqrt{M^2-1}}, \qquad m = \frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1}, \tag{24}$$

and $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ are the parameters arising in the finite gap integration method of the KdV equation (see [15]).

It is instructive to establish a direct asymptotic correspondence between fully nonlinear periodic solution (12) of the GP equation and its small-amplitude KdV counterpart (23). For that, we first represent the arbitrary parameters p_1, p_2, p_3 in the form of asymptotic expansions in the small parameter ε,

$$p_i = 1 + \varepsilon p_i^{(1)} + \cdots, \qquad i = 1, 2, 3, \tag{25}$$

and then substitute the expansions (19), (21), (25) into the periodic solution (12) to obtain the same solution (23) but parameterized by $p_i^{(1)}$ so that comparison with (23) yields

$$\begin{aligned} p_1 &= 1 - \tfrac{1}{2}M^2\varepsilon(\lambda_3 - \lambda_1 - \lambda_2), \\ p_2 &= 1 - \tfrac{1}{2}M^2\varepsilon(\lambda_2 - \lambda_1 - \lambda_3), \\ p_3 &= 1 - \tfrac{1}{2}M^2\varepsilon(\lambda_1 - \lambda_2 - \lambda_3). \end{aligned} \tag{26}$$

Then, using analogous asymptotic expansion for a,

$$a = a^{(0)} + \varepsilon a^{(1)} + \cdots, \tag{27}$$

relation (7) and the last relation in (14), we recover the asymptotic expression (24) for the slope. Inverse expressions for λ_js in terms of p_js, following from (26) are

$$\lambda_1 = \frac{p_1 + p_2 - 2}{M^2\varepsilon}, \qquad \lambda_2 = \frac{p_1 + p_3 - 2}{M^2\varepsilon}, \qquad \lambda_3 = \frac{p_2 + p_3 - 2}{M^2\varepsilon}. \tag{28}$$

The soliton solution of the KdV equation corresponds to $m = 1$ in (23). In terms of the original parameters p_j, we have $p_2 = p_3 = 1$ (see (13), (26)), which implies by (28) that $\lambda_1 = \lambda_2 \equiv \lambda$ and $\lambda_3 = 0$. Then

$$p_1 = 1 + M^2\varepsilon\lambda \tag{29}$$

and, since $p_1 \leqslant p_2 \leqslant p_3$ (see (11)), we have $p_1 \leqslant 1$ and thus $\lambda \leqslant 0$. Now, from (23) we have for $m = 1$ the small-amplitude dark soliton profile

$$n = 1 - \frac{-M^2\varepsilon\lambda}{\cosh^2\left[\sqrt{-\varepsilon\lambda}(x - ay)\right]}, \tag{30}$$

and from (24) its slope is

$$a = \sqrt{M^2 - 1} - \frac{\varepsilon\lambda M^2}{2\sqrt{M^2 - 1}}. \tag{31}$$

Since $\lambda \leqslant 0$, one can see from (31) that $a \geqslant a_M$, where $a_M = \sqrt{M^2 - 1}$, i.e. the shallow (KdV) dark solitons always lie within (and close to) the Mach cone.

If we introduce the inverse half-width κ of the soliton according to

$$\kappa = 2\sqrt{-\varepsilon\lambda}, \tag{32}$$

then (30) assumes a more conventional form

$$n = 1 - \frac{M^2\kappa^2}{4\cosh^2[\kappa(x - ay)/2]}, \tag{33}$$

which also follows directly from the oblique dark soliton solution (17) in the small-amplitude limit [17].

Thus, we have established an asymptotic correspondence between the stationary periodic two-dimensional solution of the GP equation characterized by four independent parameters and the familiar three-parameter cnoidal wave solution of the KdV equation.

4. Periodic stationary waves with large slopes in highly supersonic flow

If the flow is highly supersonic ($M \gg 1$), then the system (5) can be reduced to the NLS equation [10]. Indeed, after introduction of the new variables

$$u = M + u_1 + O(1/M), \qquad T = x/2M, \qquad Y = y, \tag{34}$$

where $u_1 \to 0$ as $|Y| \to \infty$, we arrive, to leading order in M^{-1}, at the system

$$\begin{aligned} &\frac{1}{2}n_T + (nv)_Y = 0 \\ &\frac{1}{2}v_T + vv_Y + n_Y + \left(\frac{n_Y^2}{8n^2} - \frac{n_{YY}}{4n}\right)_Y = 0 \end{aligned} \tag{35}$$

and the equation

$$\tfrac{1}{2}u_{1T} + vu_{1Y} = 0. \tag{36}$$

The leading term of the highly supersonic expansion of the potentiality condition (15) together with equation (36) implies $u_1 \equiv 0$. The decoupled from (36) equations (35) represent the hydrodynamic form of the 1D NLS equation

$$\mathrm{i}\Psi_T + \Psi_{YY} - 2|\Psi|^2\Psi = 0 \tag{37}$$

for a complex field variable

$$\Psi = \sqrt{n}\exp\left(\mathrm{i}\int^Y v(Y', T)\,\mathrm{d}Y'\right). \tag{38}$$

Note that this NLS equation, unlike the exact (1+1) reduction (2) of the GP equation, represents a (0+2) asymptotic *approximation* and, in addition, contains a completely different, compared to (2), set of independent variables. Periodic solution of the NLS equation (37) is well known (see, e.g., [15]) and for the density $n = |\Psi|^2$ can be written in the form

$$n = \nu_1 + (\nu_2 - \nu_1)\mathrm{sn}^2\left[\sqrt{\nu_3 - \nu_1}\left(y - \frac{s_1}{2M}x\right), m\right], \tag{39}$$

where

$$s_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \qquad m = \frac{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)} \tag{40}$$

and $\nu_1 \leqslant \nu_2 \leqslant \nu_3$ are expressed in terms of the parameters $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \lambda_4$ as follows,

$$\begin{aligned} \nu_1 &= \tfrac{1}{4}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)^2, \\ \nu_2 &= \tfrac{1}{4}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)^2, \\ \nu_3 &= \tfrac{1}{4}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2, \end{aligned} \tag{41}$$

and $m = (\nu_2 - \nu_1)/(\nu_3 - \nu_1)$.

For $M \gg 1$ the stationary wave has a large slope with respect to the y-axis, that is $a \gg 1$ and the general solution (12) assumes the form

$$n = p_1 + (p_2 - p_1)\mathrm{sn}^2\big(\sqrt{p_3 - p_1}(y - x/a); m\big). \tag{42}$$

Since the asymptotic solution (42) of the GP equation is characterized by four parameters p_1, p_2, p_3, a, the correspondence with the four-parameter $(\nu_1, \nu_2, \nu_3, s_1)$ periodic solution (39) is readily established by a direct comparison:

$$p_1 = \nu_1, \qquad p_2 = \nu_2, \qquad p_3 = \nu_3, \qquad a = \frac{2M}{s_1}. \tag{43}$$

The dark soliton reduction of the NLS cnoidal wave solution is obtained by putting $\lambda_2 = \lambda_3$, i.e. $m = 1$, in (39). To get a unit pedestal for the soliton it is convenient to choose

$$\lambda_1 = -1, \qquad \lambda_4 = 1. \tag{44}$$

Then, denoting $\lambda_2 = \lambda_3 \equiv \lambda$ we get from (43), (40)

$$a = \frac{M}{\lambda}. \tag{45}$$

Thus, in the soliton limit we obtain

$$n = 1 - \frac{1-\lambda^2}{\cosh^2\left[\sqrt{1-\lambda^2}(y - x/a)\right]} \tag{46}$$

which agrees with the asymptotic representation of a highly supersonic oblique GP soliton in [17]. Without loss of generality we consider the waves in the upper half-plane ($a > 0$), then it follows from (44) that $0 \leqslant \lambda \leqslant 1$. Thus, $a > a_M$ and, therefore, the dark solitons in highly supersonic flows are always generated within the Mach cone.

If the NLS soliton is shallow, then it must be consistent with the corresponding asymptotic as $M \gg 1$ of the KdV soliton. Indeed, we introduce $\lambda^2 = 1 - M^2\kappa^2/4$, where $M\kappa \ll 1$, and obtain the KdV soliton solution (33) but now with a equal to

$$a = M/\sqrt{1 - M^2\kappa^2/4} \cong M + \tfrac{1}{8}M^3\kappa^2 \tag{47}$$

which is the approximation of equation (24) with chosen values of λ_i and $M \gg 1$.

The connection between the soliton parameters λ for the KdV and the 'shallow' NLS limits is given by the expression

$$\lambda_{\mathrm{NLS}}^2 = 1 + M^2\varepsilon\lambda_{\mathrm{KdV}}. \tag{48}$$

Since $\lambda_{\mathrm{KdV}} \leqslant 0$, we have $\lambda_{\mathrm{NLS}} \leqslant 1$ but $1 - \lambda_{\mathrm{NLS}} \ll 1$.

5. Linear waves

At last, let us consider the limit of linear waves propagating on a constant background (this case is more general than the linear wave limit within the KdV approximation (22) as it does not imply the long-wave scaling (20)). In this case equation (12) with $m \ll 1$ (i.e. $p_2 - p_1 \ll 1$) can be transformed to

$$n = 1 - \frac{1}{2}(p_2 - p_1)\cos\left[2\sqrt{\frac{p_3 - p_1}{1 + a^2}}(x - ay)\right], \tag{49}$$

where we have assumed that the mean background density is equal to unity, $(p_1 + p_2)/2 = 1$.

Further, in the limit of vanishing amplitude $p_2 \to p_1$ and large wavelength we take

$$p_1 = p_2 = 1, \qquad p_3 = 1 + M^2\varepsilon, \tag{50}$$

and obtain for $M^2\varepsilon \ll 1$,

$$a = \sqrt{M^2 - 1} - \frac{M^2\varepsilon}{2\sqrt{M^2 - 1}}. \tag{51}$$

Now we have $a < a_M$, which means that the *linear* long waves, in contrast to solitons, are always generated *outside* the Mach cone. If we denote

$$2\sqrt{\frac{p_3 - p_1}{1 + a^2}} \cong 2\sqrt{\varepsilon} = K, \qquad -\frac{1}{2}(p_2 - p_1) = Q, \tag{52}$$

then we obtain the stationary linear wave solution in the form

$$n = 1 + Q\cos\left[K\left(x - \left(\sqrt{M^2-1} - \frac{M^2K^2}{8\sqrt{M^2-1}}\right)y\right)\right], \tag{53}$$

which maps to the tail of the oblique soliton solution in the KdV limit (33) by the change $\kappa \mapsto \mathrm{i}K$.

As was noted above, the general formula (49) describes linear waves of an arbitrary wavelength and it can be mapped to the tails of the general soliton solution (17).

6. Conclusions

We have obtained the family of exact fully nonlinear stationary periodic solutions of the 2D Gross–Pitaevskii equation and studied in detail their particular asymptotic reductions corresponding to the solutions of the KdV and NLS equations.

The obtained solutions provide a basis for further studies connected with the description of dispersive shock waves observed in recent experiments [2] of the flow of a BEC past obstacles as well as in numerical simulations [17]. Some straightforward implications about the characteristic features of the wave patterns arising in the flow of a BEC past obstacle have been made from the expressions for the slope a in the obtained asymptotic reductions of the full periodic solution. The deep solitons asymptotically described by the NLS equation have large slopes, while the shallow solitons obey the KdV equation and have slopes close the Mach cone, $a_M = \sqrt{M^2-1}$, and they are located inside the Mach cone. The linear wave packets are always located outside the Mach cone. Detailed theories of these waves patterns generated by the flow of a BEC past obstacles will be developed elsewhere.

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